# METHODS OF ATTACK OF ORIGINALS IN GEOMETRY[1]

APART from all considerations of method, two things are necessary for satisfactory work in the solution of "original" propositions in geometry. First, a thorough knowledge of fundamental definitions and propositions. If the proof of my proposition depends upon the fact that two triangles are congruent, I must be able to recognize the relation without referring to my book, or it is likely to escape my notice. I must know, further, exactly how to prove triangles congruent. To this end, it is well to generalize theorems. The theorems concerning congruent triangles are all contained in the general theorem: Two triangles are congruent if any three independent parts of the one are equal to the corresponding parts of the other, except in the ambiguous case. It is well to make a summary of theorems which are of constant use, but cannot be united in a general theorem. We are often called upon, for example, to prove angles equal in the course of a demonstration; the methods of doing this must be at our fingers' ends, and some such summary as the following may help:

TESTS FOR EQUAL ANGLES.

1. By construction.
2. Vertical angles.
3. Supplements or complements of equal angles.
4. Corresponding angles in congruent or similar triangles.
5. Opposite equal sides in the same triangle.
6. Made by parallel lines,
7. Central or inscribed angles on the same arc, etc.

The second requisite for good work is a geometric imagination. This is necessary to see relations, even when all needed lines are drawn; it is even more necessary to determine what

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extra lines are needed. I believe it is absolutely useless for one to work on "original" propositions who has no geometric imagination. But the trouble is more likely to be an undeveloped imagination; it needs exercise, such exercise as may be obtained from a frequent application of the principle of continuity.

As to methods proper: We are always given a definite thing to prove or a definite thing to do. We are required to prove, for example, that the angles opposite the equal sides of an isosceles triangle are equal. There are a great number of things which are true of an isosceles triangle; we want a particular one of those truths. How can we avoid, for the time being, those which we are not seeking and go straight to the one we want? The only way to do this is to begin at the conclusion, to assume this either true or false, and reason back to known facts. This method is not confined to geometry; it is the method of every science and of everyday life. This is the one and only method of attacking an "original" in geometry, the method of analysis. Euclid defined it as follows: "Analysis is the obtaining of the thing sought by assuming it and reasoning up to an admitted truth; synthesis is the obtaining of the thing sought by reasoning up to the inference and proof of it." The *reductio ad absurdum* would not come under this definition, but there is no difference in the essence of the method. I shall include it under the head of analysis, and extend Euclid's definition to read: "Analysis is the obtaining of the thing sought by either assuming it true or false, and reasoning up to an admitted truth or falsehood." Analysis, as defined by Euclid, is not conclusive unless the converse of every proposition involved is true. To remove all doubt, the Greeks generally added a synthesis. The modern form of analysis, the method of successive substitutions, so states the case as to avoid the necessity that the reasoning be reversible. We reason as follows: We wish to prove A true; if B is true, it is true; if C is true, B is true; but C is true, therefore A is true.

I feel so strongly the importance of this method of analysis that I trust a slight digression may be pardoned. I know of no text which in any way emphasizes the method of analysis in the

treatment of its text demonstrations. I believe the analysis should be openly recognized as preceding the synthesis. I make a plea for it as a matter of teaching. When we insist upon students knowing exactly what they are to do, when we ask them to bear in mind what they are trying to prove, we are really asking them, in their own minds at least, to start at the conclusion and reason backwards. The brighter part of a class instinctively and unconsciously uses the method of analysis, and, if there is any philosophy in teaching, it should bring out into the open this process which is now so largely simply instinctive and unconscious. I do not wonder that demonstrations are committed to memory. Even when the reasoning is followed and the student sees that the proposition is proved, the steps may seem perfectly arbitrary and be a mere matter of memory. I remember that, as a student, I greatly admired the man who first inscribed a regular pentagon in a circle. How he ever thought of so improbable a thing as dividing a line into extreme and mean ratio was more than I could see. I learned as a perfectly arbitrary fact that to construct a regular pentagon it was necessary to divide a line into extreme and mean ratio. Let us look at this proposition from the other side; let us start with the regular pentagon. If we draw the diagonals, we shall have a number of isosceles triangles, whose equal angles are each double their third angle. We can readily see how to construct the regular pentagon if we could once get this triangle. Our problem is now reduced to constructing an isosceles triangle such that each of its equal angles is double the third angle. We now suppose that we actually have such a triangle and look for a relation between the lengths of its sides; we are led straight to the conclusion that, if one of the equal sides is divided into extreme and mean ratio, the larger segment is equal to the base of our triangle. This is no longer an arbitrary process, it is no longer a matter of happening to turn out right. I know why the old Greek divided his line into extreme and mean ratio. I welcome the introduction of "original" propositions into our courses, because it is forcing this powerful method of analysis

into clear recognition. As a means of mental training, as a method of discovery, the method of analysis stands preëminent; the method of synthesis excels simply as a means of exposition and persuasion.

When we are given a problem in elementary plane geometry (I confine this analysis to plane geometry to save time), we are asked to determine one or more points, one or more straight lines, or one or more circles. We have a straight edge and compasses to work with. To draw a straight line with a straight edge, we must have two points. To draw a circle with a pair of compasses, we must have one point for its center and two points the distance between which will give us the radius. In the last analysis, then, the solution of any problem in plane geometry resolves itself into the determination of one or more points. How can we locate points on a plane? For example, how can we locate a point on this page? We might locate it by saying it is six inches from the top and two inches from the right-hand edge. If it is six inches from the top, it is somewhere on a line parallel with the top and six inches from it. If it is two inches from the right-hand edge, it is somewhere on a line parallel with that edge and two inches from it. If it fulfills both conditions, it must be at the intersection of these lines. We have located our point by the intersection of loci, and I venture the assertion that our only method of locating a point is by the intersection of loci. The solution of every problem involves directly or indirectly the intersection of loci. For example, we are asked to bisect an angle. With the vertex as a center and with any radius we describe a circumference (a locus); this intersects the two arms of the angle (other loci) and determines two points. With these points as centers and with the same radius we describe two circumferences (loci); these intersect and determine points, etc. It should be remarked that this method of intersecting loci is not a different method from that of analysis. It is simply a device or submethod. We must use analysis to find what loci we want.

We are sometimes asked to find the locus of a point fulfilling

a given condition, and in order to apply the method of analysis we must first know what the locus is and then find our proof. Suppose we have no idea of what the locus is. Our first object is to find the shape of the locus. We must make a careful drawing on paper, determining a number of points on the locus, and then join the points by a smooth curve. We can now guess what the curve is and apply the method of analysis, noticing carefully any peculiar points, for they are liable to throw light on the demonstration.

There are other submethods which analysis sometimes suggests. Such are the application of symmetry, superposition, folding, parallel translation, and turning of part of a figure through an angle. The essential point in all of these, except the application of symmetry, is the fact that a figure or part of a figure may be moved about in any way we please. We can subject such part of our figure as we see fit to a simple translation, a simple rotation, or a combined translation or rotation. I think it is better to express this in its general form. In the application of these devices the geometric imagination may find ample range. A happy inspiration will now and then suggest a simply elegant proof.

In conclusion, the one general method of attack is the method of analysis. But mere method is useless; it must be accompanied by a geometric imagination and a thorough knowledge of fundamental definitions and demonstration.

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